Exercise 5F

1 **a** l_1 is the line $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

l_2 is the line $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$$\begin{pmatrix}1\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}-1\\-3\\2\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$1 + \lambda = -1 + \mu$ **(1)**

$3 - \lambda = -3 + \mu$ **(2)**

$5\lambda = 2 + 2\mu$ **(3)**

Adding **(1)** and **(2)** gives:

$4 = -4 + 2\mu \Rightarrow \mu = 4$

Substituting $\mu = 4$ into **(1)** gives:

$1 + \lambda = -1 + 4 \Rightarrow \lambda = 2$

Substituting $\lambda = 2$ and $\mu = 4$ into **(3)** gives:

$5(2) = 2 + 2(4)$

$10 = 10$

Substituting $\lambda = 2$ into $\mathbf{r} = \begin{pmatrix}1\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\5\end{pmatrix}$ gives:

$$\mathbf{r} = \begin{pmatrix}1\\3\\0\end{pmatrix} + 2\begin{pmatrix}1\\-1\\5\end{pmatrix}$$

$$= \begin{pmatrix}3\\1\\10\end{pmatrix}$$

Therefore, l_1 and l_2 intersect at the point (3, 1, 10)

1 b l_1 is the line $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

l_2 is the line $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k})$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$3 + \lambda = 4 - \mu$ **(1)**

$2 + \lambda = 3 + \mu$ **(2)**

$1 + 2\lambda = -\mu$ **(3)**

Adding **(1)** and **(2)** gives:

$5 + 2\lambda = 7 \Rightarrow \lambda = 1$

Substituting $\lambda = 1$ into **(1)** gives:

$3 + 1 = 4 - \mu \Rightarrow \mu = 0$

Substituting $\lambda = 1$ and $\mu = 0$ into **(3)** gives:

$1 + 2(1) = 0$

$3 \neq 0$

Therefore, l_1 and l_2 do not intersect.

1 c l_1 is the line $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{j} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

l_2 is the line $\mathbf{r} = \mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{5}{2}\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

$$\begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}1\\5/2\\5/2\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$1 + 2\lambda = 1 + \mu$ **(1)**

$3 + 3\lambda = \frac{5}{2} + \mu$ **(2)**

$5 + \lambda = \frac{5}{2} - 2\mu$ **(3)**

Subtracting **(1)** from **(2)** gives:

$2 + \lambda = \frac{3}{2} \Rightarrow \lambda = -\frac{1}{2}$

Substituting $\lambda = -\frac{1}{2}$ into **(1)** gives:

$1 + 2\left(-\frac{1}{2}\right) = 1 + \mu \Rightarrow \mu = -1$

Substituting $\lambda = -\frac{1}{2}$ and $\mu = -1$ into **(3)** gives:

$5 - \frac{1}{2} = \frac{5}{2} - 2(-1)$

$\frac{9}{2} = \frac{9}{2}$

Substituting $\lambda = -\frac{1}{2}$ into $\mathbf{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ gives:

$$\mathbf{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\3/2\\9/2\end{pmatrix}$$

Therefore, l_1 and l_2 intersect at the point $\left(0, \frac{3}{2}, \frac{9}{2}\right)$

2 a $l: \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

$\Pi: \mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 16$

If the line and the plane meet then:

$$\begin{pmatrix} 1-2\lambda \\ 1+\lambda \\ 1-4\lambda \end{pmatrix}\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 16$$

$3(1-2\lambda) - 4(1+\lambda) + 2(1-4\lambda) = 16$

$3 - 6\lambda - 4 - 4\lambda - 2 - 8\lambda = 16$

$1 - 18\lambda = 16$

$18\lambda = -15$

$\lambda = -\frac{5}{6}$

Therefore, the line and the plane meet at the point:

$$\begin{pmatrix} 1-2\left(-\frac{5}{6}\right) \\ 1+\left(-\frac{5}{6}\right) \\ 1-4\left(-\frac{5}{6}\right) \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{1}{6} \\ \frac{13}{3} \end{pmatrix}$$

b $l: \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\Pi: \mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1$

If the line and the plane meet then:

$$\begin{pmatrix} 2+\lambda \\ 3+\lambda \\ -2+\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1$$

$(2+\lambda) + (3+\lambda) - 2(-2+\lambda) = 1$

$2 + \lambda + 3 + \lambda + 4 - 2\lambda = 1$

$9 = 1$

Therefore, the line and the plane do not meet.

2 c $l : \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$

$\Pi : \mathbf{r}.(3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

If the line and the plane meet then:

$$\begin{pmatrix} 1 \\ 1+2\lambda \\ 1-2\lambda \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} = 1$$

$$3 - (1+2\lambda) - 6(1-2\lambda) = 1$$

$$3 - 1 - 2\lambda - 6 + 12\lambda = 1$$

$$-4 + 10\lambda = 1$$

$$10\lambda = 5$$

$$\lambda = \frac{1}{2}$$

Therefore, the line and the plane meet at the point:

$$\begin{pmatrix} 1 \\ 1+2\left(\frac{1}{2}\right) \\ 1-2\left(\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

3 a $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ is normal to Π_1 and $4\mathbf{i}-\mathbf{j}-2\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 4 & -1 & -2 \end{vmatrix} = 3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$$

The Cartesian equation of both planes are:

Π_1: $3x-2y-z=5$ **(1)**

Π_2: $4x-y-2z=5$ **(2)**

Setting $y=0$ and then multiplying equation **(1)** by 2 and subtracting equation **(2)** gives

$2x=5 \Rightarrow x=\frac{5}{2}$

Substituting for x in equation **(2)** gives

$4\times\frac{5}{2}-2z=5 \Rightarrow z=\frac{5}{2}$

So $\frac{5}{2}\mathbf{i}+\frac{5}{2}\mathbf{k}$ is a point on the line

An equation passing though a point with position vector **a** and parallel to vector **b** has the vector equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, so the equation of the line of intersection of the two planes is

$\mathbf{r}=\frac{5}{2}\mathbf{i}+\frac{5}{2}\mathbf{k}+\lambda(3\mathbf{i}+2\mathbf{j}+5\mathbf{k})$

b $5\mathbf{i}-\mathbf{j}-2\mathbf{k}$ is normal to Π_1 and $16\mathbf{i}-5\mathbf{j}-4\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & -2 \\ 16 & -5 & -4 \end{vmatrix} = -6\mathbf{i}-12\mathbf{j}-9\mathbf{k}$$

Dividing by the scalar –3 to get this vector in a simple form, gives the direction of the line as $2\mathbf{i}+4\mathbf{j}+3\mathbf{k}$

The Cartesian equation of both planes are:

Π_1: $5x-y-2z=16$ **(1)**

Π_2: $16x-5y-4z=53$ **(2)**

Setting $z=0$ and then multiplying equation **(1)** by 5 and subtracting equation **(2)** gives

$(25-16)x=80-53 \Rightarrow 9x=27 \Rightarrow x=3$

Substituting for x in equation **(1)** gives

$5\times3-y=16 \Rightarrow y=-1$

So $3\mathbf{i}-\mathbf{j}$ is a point on the line

The equation of the line of intersection of the two planes is

$\mathbf{r}=3\mathbf{i}-\mathbf{j}+\lambda(2\mathbf{i}+4\mathbf{j}+3\mathbf{k})$

3 c $\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is normal to Π_1 and $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 4 & -3 & -2 \end{vmatrix} = 9\mathbf{i}+6\mathbf{j}+9\mathbf{k}$$

Dividing by the scalar 3 to get this vector in a simple form, gives the direction of the line as $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

The Cartesian equation of both planes are:

$\Pi_1: \ x-3y+z=10$ **(1)**

$\Pi_2: \ 4x-3y-2z=1$ **(2)**

Setting $z = 0$ and then subtracting equation **(2)** from equation **(1)** gives

$3x=-9 \Rightarrow x=-3$

Substituting for x in equation **(1)** gives

$-3-3y=10 \Rightarrow y=-\frac{13}{3}$

So $-3\mathbf{i}-\frac{13}{3}\mathbf{j}$ is a point on the line

The equation of the line of intersection of the two planes is

$\mathbf{r}=-3\mathbf{i}-\frac{13}{3}\mathbf{j}+\lambda(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

4 $\Pi_1: \mathbf{r}.(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=1$ and $\Pi_2: \mathbf{r}.(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})=7$

The normal of the planes are in the directions:

$\mathbf{n}_1=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $\mathbf{n}_2=-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$

Let the angle between these normal be θ, where:

$$\cos\theta=\frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k}).(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{\sqrt{1^2+2^2+(-2)^2}\times\sqrt{(-4)^2+4^2+7^2}}$$

$$=\frac{-4+8-14}{3\times 9}$$

$$=-\frac{10}{27}$$

$\theta=111.7...$

So the acute angle between the planes is:

$180-111.7...=68.26...$

$=68.3°$ (3 s.f.)

5 $\Pi_1 : \mathbf{r}.(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=9$ and $\Pi_2 : \mathbf{r}.(5\mathbf{i}-12\mathbf{k})=7$

The normal of the planes are in the directions:

$\mathbf{n}_1 = 3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $\mathbf{n}_2 = 5\mathbf{i}-12\mathbf{k}$

Let the angle between these normal be θ, where:

$$\cos\theta = \frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k}).(5\mathbf{i}-12\mathbf{k})}{\sqrt{3^2+(-4)^2+12^2}\times\sqrt{5^2+(-12)^2}}$$

$$= \frac{15-144}{13\times13}$$

$$= -\frac{129}{169}$$

$\theta = 139.752...$

So the acute angle between the planes is:

$180-139.752... = 40.24...$

$= 40.2°$ (3 s.f.)

6 $l : \mathbf{r} = 2\mathbf{i}+\mathbf{j}-5\mathbf{k}+\lambda(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})$

$\Pi : \mathbf{r}.(2\mathbf{i}+\mathbf{j}-2\mathbf{k})=13$

The normal to the plane is in the direction:

$\mathbf{n} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

Let the angle between this normal and the line, l, be θ, where:

$$\cos\theta = \frac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k}).(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\times\sqrt{2^2+1^2+(-2)^2}}$$

$$= \frac{8+4-14}{9\times3}$$

$$= -\frac{2}{27}$$

$\theta = 94.24...$

The acute angle, α, between the line and the plane is found using $\alpha+\theta = 90°$

Hence:

$\alpha + 94.248\ldots = 90$

$\alpha = -4.248\ldots$

So the acute angle between the planes is 4.25° (3 s.f.)

7 $l:\mathbf{r}=-\mathbf{i}-7\mathbf{j}-13\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{j}-12\mathbf{k})$

$\Pi:\mathbf{r}.(4\mathbf{i}-4\mathbf{j}-7\mathbf{k})=9$

The normal to the plane is in the direction:

$\mathbf{n}=4\mathbf{i}-4\mathbf{j}-7\mathbf{k}$

Let the angle between this normal and the line, l, be θ, where:

$$\cos\theta=\frac{(3\mathbf{i}+4\mathbf{j}-12\mathbf{k}).(4\mathbf{i}-4\mathbf{j}-7\mathbf{k})}{\sqrt{3^2+4^2+(-12)^2}\times\sqrt{4^2+(-4)^2+(-7)^2}}$$

$$=\frac{12-16+84}{13\times 9}$$

$$=\frac{80}{117}$$

$\theta=46.86...$

The acute angle, α, between the line and the plane is found using $\alpha+\theta=90°$

Hence:

$\alpha+46.86\ldots=90$

$\alpha=43.13\ldots$

So the acute angle between the planes is 43.1° (3 s.f.)

8 First find a normal **n** to the plane

$$\mathbf{n}=(4\mathbf{i}-\mathbf{j}-\mathbf{k})\times(4\mathbf{i}-5\mathbf{j}-3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -1 & -1\\ 4 & -5 & 3\end{vmatrix}=-8\mathbf{i}-16\mathbf{j}-16\mathbf{k}$$

Dividing by 8, this simplifies to $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

Let θ be the angle between the line and the normal to the plane.

Using the definition of the scalar product $\mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$ gives

$$\cos\theta=\frac{(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k}).(-\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{(-4)^2+(-7)^2+4^2}\sqrt{(-1)^2+(-2)^2+(-2)^2}}=\frac{4+14-8}{\sqrt{81}\times\sqrt{9}}=\frac{10}{9\times 3}=\frac{10}{27}$$

Let α be the acute angle between the line and the plane. As $\cos\theta>0$, θ is acute and so $\theta+\alpha=90°$

$$\cos\theta=\frac{10}{27}\Rightarrow\theta=68.3°\ \text{(3 s.f.)},\ \text{so}\ \alpha=90°-68.3°=21.7°\ \text{(3 s.f.)}$$

9 a $\Pi:\mathbf{r}.(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=81$

The unit vector parallel to $10\mathbf{i}+10\mathbf{j}+23\mathbf{k}$ is:

$$\hat{\mathbf{n}}=\frac{1}{\sqrt{10^2+10^2+23^2}}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

$$=\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

So the equation of the plane may be written as: $\mathbf{r}.\hat{\mathbf{n}}=\frac{81}{27}$

$=3$

Therefore, the perpendicular distance from the origin to the plane is 3.

9 b The plane, Π', parallel to Π, passing through the point (−1, −1, 4) has the equation:

$$\mathbf{r}\cdot\hat{\mathbf{n}} = (-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= -\frac{10}{27}-\frac{10}{27}+\frac{4\times 23}{27}$$
$$= \frac{8}{3}$$

So the perpendicular distance from (−1, −1, 4) to the plane is $\frac{8}{3}$

and the distance between Π and Π' is:

$$3-\frac{8}{3}=\frac{1}{3}$$

c The plane, Π', parallel to Π, passing through the point (2, 1, 3) has the equation:

$$\mathbf{r}\cdot\hat{\mathbf{n}} = (2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{20}{27}+\frac{10}{27}+\frac{3\times 23}{27}$$
$$= \frac{11}{3}$$

So the perpendicular distance from (2, 1, 3) to the plane is $\frac{11}{3}$

and the distance between Π and Π' is:

$$3-\frac{11}{3}=-\frac{2}{3}$$
$$=\frac{2}{3}$$

d The plane, Π', parallel to Π, passing through the point (6, 12, −9) has the equation:

$$\mathbf{r}\cdot\hat{\mathbf{n}} = (6\mathbf{i}+12\mathbf{j}-9\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{60}{27}+\frac{120}{27}-\frac{9\times 23}{27}$$
$$= -1$$

So the perpendicular distance from (6, 12, −9) to the plane is −1
and the distance between Π and Π' is:

$$3-(-1)=4$$

10 a $\Pi_1 : \mathbf{r}.(6\mathbf{i}+6\mathbf{j}-7\mathbf{k}) = 55$ and $\Pi_2 : \mathbf{r}.(6\mathbf{i}+6\mathbf{j}-7\mathbf{k}) = 22$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{6^2+6^2+(-7)^2}}(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})$$
$$= \frac{1}{11}(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})$$

Hence for Π_1:

$$\mathbf{r}\cdot\hat{\mathbf{n}} = \frac{55}{11} = 5$$

and for Π_2:

$$\mathbf{r}\cdot\hat{\mathbf{n}} = \frac{22}{11} = 2$$

Since the planes are parallel, the distance between them is:

$5 - 2 = 3$

b $\Pi_1 : \mathbf{r} = 3\mathbf{i}+4\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{k})+\mu(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$

$\Pi_2 : \mathbf{r} = 14\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{j}+\mathbf{k})+\mu(8\mathbf{i}-9\mathbf{j}-\mathbf{k})$

As the planes are parallel, they share the same unit normal. Calculate it using Π_1:

$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 8 & 3 & 3 \end{vmatrix}$$
$$= \mathbf{i}(0-3)-\mathbf{j}(12-8)+\mathbf{k}(12-0)$$
$$= -3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{(-3)^2+(-4)^2+12^2}}(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$
$$= \frac{1}{13}(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$

For Π_1:

$$\mathbf{r}.\hat{\mathbf{n}} = \frac{1}{13}(3\mathbf{i}+4\mathbf{j}-\mathbf{k}).(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$
$$= -\frac{13}{13}$$
$$= -1$$

And for Π_2:

$$\mathbf{r}.\hat{\mathbf{n}} = \frac{1}{13}(14\mathbf{i}+2\mathbf{j}+2\mathbf{k}).(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$
$$= -\frac{26}{13}$$
$$= -2$$

Since the planes are parallel, the distance between them is:

$|(-1)-(-2)| = 1$

11 The shortest distance between two skew lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ is $\left|\frac{(a-c).(b\times d)}{|b\times d|}\right|$

In this case: $a - c = \mathbf{i} - (3\mathbf{i} - \mathbf{j} + \mathbf{k}) = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$b\times d = (-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k})\times(2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -12 & 11 \\ 2 & 6 & -5 \end{vmatrix} = -6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\text{Shortest distance} = \left|\frac{(-2\mathbf{i} + \mathbf{j} - \mathbf{k}).(-6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})}{\sqrt{(-6)^2 + 7^2 + 6^2}}\right| = \left|\frac{12 + 7 - 6}{\sqrt{121}}\right| = \frac{13}{11}$$

12 $l_1 : \mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$

$l_2 : \mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$

Let A be a general point on l_1 and let B be a general point on l_2, then:

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} \text{ where } t = \mu - \lambda$$

Let the distance AB be x, then:

$$x^2 = (-2-3t)^2 + (2-4t)^2 + (0-5t)^2$$
$$= 4 + 12t + 9t^2 + 4 - 16t + 16t^2 + 25t^2$$
$$= 50t^2 - 4t + 8$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2) = 100t - 4$$

At the minimum:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2) = 0$$

Therefore:

$$100t - 4 = 0 \Rightarrow t = \frac{1}{25}$$

Hence:

$$x^2 = 50\left(\frac{1}{25}\right)^2 - 4\left(\frac{1}{25}\right) + 8$$

$$x^2 = \frac{198}{25}$$

$$x = \frac{\sqrt{198}}{25}$$

13 a $l_1 : \mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

$l_2 : \mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

If the lines meet then:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

$1 + 2\lambda = -1 + 2\mu$ **(1)**

$1 - \lambda = 1 - 5\mu$ **(2)**

$5\lambda = 2 + \mu$ **(3)**

Adding 2 × **(2) to (1)** gives:

$3 = 1 - 8\mu \Rightarrow \mu = -\frac{1}{4}$

Substituting $\mu = -\frac{1}{4}$ into **(1)** gives:

$1 + 2\lambda = -1 + 2\left(-\frac{1}{4}\right)$

$2\lambda = -2 - \frac{1}{2}$

$\lambda = -\frac{5}{4}$

Substituting $\mu = -\frac{1}{4}$ and $\lambda = -\frac{5}{4}$ into **(3)** gives:

$5\left(-\frac{5}{4}\right) = 2 - \frac{1}{4}$

$-\frac{25}{4} = \frac{7}{4}$

Therefore, the lines do not meet.

Consider that l_1 and l_2 are of the form:

$l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $l_2 : \mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$

Then the shortest distance between the lines is given by:

$$d = \frac{(\mathbf{a} - \mathbf{c})\cdot(\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|}$$

$\mathbf{a} - \mathbf{c} = \mathbf{i} + \mathbf{j} - (-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$= 2\mathbf{i} - 2\mathbf{k}$

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 5\\ 2 & -5 & 1\end{vmatrix}$$

$= \mathbf{i}(-1 + 25) - \mathbf{j}(2 - 10) + \mathbf{k}(-10 + 2)$

$= 24\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$

Therefore:

$$d=\frac{(2\mathbf{i}-2\mathbf{k})\cdot(24\mathbf{i}+8\mathbf{j}-8\mathbf{k})}{\sqrt{24^2+8^2+(-8)^2}}$$

$$=\frac{48+0+16}{8\sqrt{3^2+1^2+(-1)^2}}$$

$$=\frac{8\sqrt{11}}{11}$$

13 b $l_1:\mathbf{r}=2\mathbf{i}+\mathbf{j}+-2\mathbf{k}+\lambda(2\mathbf{i}-2\mathbf{j}+2\mathbf{k})$

$l_2:\mathbf{r}=\mathbf{i}-\mathbf{j}+3\mathbf{k}+\mu(\mathbf{i}-\mathbf{j}+\mathbf{k})$

$2\mathbf{i}-2\mathbf{j}+2\mathbf{k}=2(\mathbf{i}-\mathbf{j}+\mathbf{k})$

Therefore, the lines are parallel.

Let A be a general point on l_1 and let B be a general point on l_2, then:

$$\overrightarrow{BA}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+t\begin{pmatrix}1\\-1\\1\end{pmatrix} \text{ where } t=\lambda-\mu$$

Let the distance AB be x, then:

$$x^2=(-1+t)^2+(-2-t)^2+(5+t)^2$$

$$=1-2t+t^2+4+4t+t^2+25+10t+t^2$$

$$=3t^2+12t+30$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2)=6t+12$$

At the minimum:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2)=0$$

Therefore:

$6t+12=0\Rightarrow t=-2$

Hence:

$$x^2=3(-2)^2+12(-2)+30$$

$$x^2=18$$

$$x=3\sqrt{2}$$

13 c $l_1: \mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

$l_2: \mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$

If the lines meet then:

$$\begin{pmatrix}1\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$1 + 2\lambda = -1 + \mu$ **(1)**

$1 + \lambda = -1 + \mu$ **(2)**

$5 - 2\lambda = 2 + \mu$ **(3)**

Subtracting **(2)** from **(1)** gives:

$\lambda = 0$

Substituting $\lambda = 0$ into **(2)** gives:

$\mu = 2$

Substituting $\lambda = 0$ and $\mu = 2$ into **(3)** gives:

$5 - 2(0) = 2 + (2)$

$5 = 4$

Therefore, the lines do not meet.

Consider that l_1 and l_2 are of the form:

$l_1: \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $l_2: \mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$

Then the shortest distance between the lines is given by:

$$d = \frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|}$$

$$\mathbf{a} - \mathbf{c} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} - (-\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 1 & -2\\1 & 1 & 1\end{vmatrix}$$
$$= \mathbf{i}(1+2) - \mathbf{j}(2+2) + \mathbf{k}(2-1)$$
$$= 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

Therefore:

$$d = \frac{(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} + \mathbf{k})}{\sqrt{3^2 + (-4)^2 + 1^2}}$$
$$= \frac{6 - 8 + 3}{\sqrt{26}}$$
$$= \frac{\sqrt{26}}{26}$$

14 $l_1: \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k})$

A is the point (4, 1, −1)

Let B be a general point on l, then:

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Let the distance AB be x, then:

$$x^2 = (-1+2\mu)^2 + (-2-\mu)^2 + (3-\mu)^2$$
$$= 14 - 6\mu + 6\mu^2$$
$$\frac{d}{d\mu}(x^2) = -6 + 12\mu$$

At the minimum:

$$\frac{d}{d\mu}(x^2) = 0$$

Therefore:

$$-6 + 12\mu = 0 \Rightarrow \mu = \frac{1}{2}$$

Hence:

$$x^2 = 14 - 6\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2$$
$$x^2 = \frac{25}{2}$$
$$x = \frac{5\sqrt{2}}{2}$$

15 a $\Pi: \mathbf{r}.(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 4$

The line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ passes through the point (2, 3, 1).
The point (2, 3, 1) also lies on the plane $\mathbf{r}.(-\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4$ as $2 \times 1 + 3 \times 1 + 1 \times (-1) = 4$
So the line and the plane have this point in common.
The line is in the direction $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which is parallel to the plane as it is perpendicular to the normal $\mathbf{i} + \mathbf{j} - \mathbf{k}$ as $-1 \times 1 + 2 \times 1 + 1 \times -1 = 0$
As the line also has a common point with the plane, it lies in the plane.

15 b $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

The line is in the direction of $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which is parallel to the plane as it is parallel to the line discussed in part **a**.

$A = (2, 3, 1)$ lies on the plane.

Let B be a general point on l, then:

$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Let the distance AB be x, then:

$$x^2 = (-3-\lambda)^2 + (-1+2\lambda)^2 + (3+\lambda)^2$$
$$= 9 + 6\lambda + \lambda^2 + 1 - 4\lambda + 4\lambda^2 + 9 + 6\lambda + \lambda^2$$
$$= 19 + 8\lambda + 6\lambda^2$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(x^2) = 18 + 12\lambda$$

At the minimum:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(x^2) = 0$$

Therefore:

$$18 + 12\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$$

Hence:

$$x^2 = 19 + 8\left(-\frac{2}{3}\right) + 6\left(-\frac{2}{3}\right)^2$$

$$x^2 = \frac{49}{3}$$

$$x = \frac{7\sqrt{3}}{3}$$